A New Method for Attitude Stabilization

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This investigation deals with motion, in a circular orbit, of a satellite consisting of a rigid body which possesses an axis of rotational symmetry and carries, on this axis, two particles that perform prescribed oscillations while the axis remains nearly normal to the plane of the orbit. Differential equations governing attitude angles are derived, stability conditions are obtained by using Floquet theory to study the boundedness of the solutions of these equations, and the practical feasibility of this method of stabilization is examined by reference to a numerical example.

IN a recent paper,[1] it was shown that the stability of steady rotations of a rigid body can be influenced by a particle forced to oscillate on a principal axis passing through the body's mass center. Because of the manner in which gravitational effects were taken into account in this analysis, the results were not directly applicable to space vehicles. However, they suggested the possibility that a similar mechanism might provide a means for attitude stabilization of such vehicles. It is the purpose of the present paper to show that this is indeed the case.

The vehicle whose motion is analyzed in the sequel consists of a rotationally symmetric rigid body and two identical particles that are constrained to oscillate in a prescribed manner on the axis of symmetry of this body. The mass center of the system is presumed to describe a circular orbit while the axis of symmetry remains nearly normal to the orbit plane.

To study the circumstances under which such motions actually are possible, differential equations governing variables that determine the angle between the axis of symmetry of the satellite and the normal to the orbit plane are derived, and the boundedness of the solutions of these equations is discussed in terms of Floquet theory.

Equations of Motion

The vehicle under consideration, shown schematically in Fig. 1, consists of a rigid body R of mass M and two particles P and P', each of mass m. P^* is the mass center of R, and C_1,C_2, and C_3 are mutually perpendicular lines, C_3 being fixed in R.

It is assumed that the moments of inertia of R with respect to C_1 and C_2 are equal to each other, from which it follows that C_1,C_2, and C_3 are principal axes of R and that the inertia properties of R are defined completely by the associated moments of inertia I_1 ($= I_2$) and I_3.

P and P' are presumed to be connected by a (light) mechanism, which can be actuated by a power supply attached to R, in such a way that the distance z (see Fig. 1) is given by

$$z = A + B \sin pt \tag{1}$$

where A,B, and p are constants and t is the time.

When P^* moves on a circular orbit, the attitude of R can be described in terms of three angles, ψ_1,ψ_2,ψ_3, generated as follows. Let O be the center of the earth (see Fig. 2) and A_1,A_2,A_3 mutually perpendicular axes with origin at P^*; A_2 points in the direction of motion of P^*, and A_3 is normal to the plane of the orbit. Perform successive right-handed rotations of amount ψ_1 about A_1, leading to B_1,B_2,B_3 (see Fig. 3); ψ_2 about B_2, leading to C_1,C_2,C_3; and ψ_3 about C_3, leading to D_1,D_2,D_3. Then, if the last set of axes is regarded as fixed in the body R, the angles ψ_1,ψ_2,ψ_3 describe the attitude of R relative to the axes system A_1,A_2,A_3. (Note that, as before, C_3 is fixed in R.)

The angle between C_3 and A_3 (A_3 is the normal to the orbit plane) depends only on ψ_1 and ψ_2 (see Fig. 3). As the sole concern of this paper will be motions during which this angle remains small, all expressions are linearized as regards ψ_1 and ψ_2 (but not ψ_3) throughout the remainder of the paper.

H, the angular momentum with respect to P^* of the nonrigid body consisting of R and the two particles P and P', and L, the gravity torque[2] for this body, both referred to the axes C_1,C_2,C_3, are given by

$$\begin{aligned} H_1 &= (I_1 + 2mz^2)(\dot{\psi}_1 - \Omega\psi_2) \\ H_2 &= (I_2 + 2mz^2)(\dot{\psi}_2 + \Omega\psi_1) \\ H_3 &= I_3(\dot{\psi}_3 + \Omega) \end{aligned}$$

and

$$\begin{aligned} L_1 &= 0 \\ L_2 &= 3\Omega^2(I_1 - I_3 + 2mz^2)\psi_2 \\ L_3 &= 0 \end{aligned}$$

where Ω is the (constant) "orbital angular speed," i.e., the angular speed of line OP^*. The angular momentum principle, applied to this nonrigid body, can be expressed as

$$\begin{aligned} L_1 &= \dot{H}_1 + \omega_2 H_3 - \omega_3 H_2 \\ L_2 &= \dot{H}_2 + \omega_1 H_1 - \omega_1 H_3 \\ L_3 &= \dot{H}_3 + \omega_1 H_2 - \omega_2 H_1 \end{aligned}$$

where $\omega_1,\omega_2,\omega_3$ describe the angular velocity of the axes system C_1,C_2,C_3 (not the angular velocity of R) and are given by

$$\begin{aligned} \omega_1 &= \dot{\psi}_1 - \Omega\psi_2 \\ \omega_2 &= \dot{\psi}_2 + \Omega\psi_1 \\ \omega_3 &= \Omega \end{aligned}$$

One thus is led to the attitude equations of motion:

$$\begin{aligned} &(I_1 + 2mz^2)\ddot{\psi}_1 + 4mz\dot{z}\dot{\psi}_1 + [(\dot{\psi}_3 + \Omega)I_3 - \\ &\quad \Omega(I_2 + 2mz^2)]\Omega\psi_1 + [(\dot{\psi}_3 + \Omega)I_3 - \\ &\quad\quad \Omega(I_1 + I_2 + 4mz^2)]\dot{\psi}_2 - 4mz\dot{z}\Omega\psi_2 = 0 \end{aligned} \tag{2}$$

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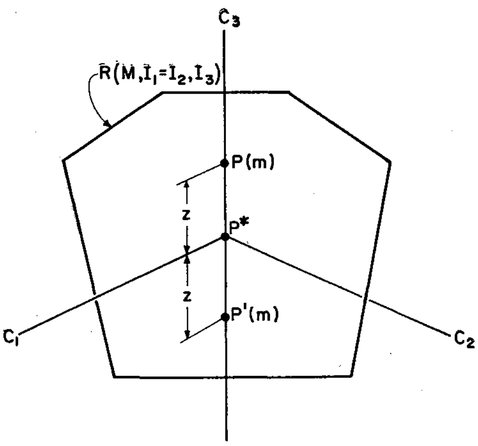

Fig. 1 Satellite vehicle

$$(I_2 + 2mz^2)\ddot{\psi}_2 + 4mz\dot{z}\dot{\psi}_2 + [(\dot{\psi}_3 + 4\Omega)I_3 - 4\Omega(I_1 + 2mz^2)]\Omega\psi_2 - [(\dot{\psi}_3 + \Omega)I_3 - \Omega(I_1 + I_2 + 4mz^2)]\dot{\psi}_1 + 4mz\dot{z}\,\Omega\psi_1 = 0 \quad (3)$$

$$\dot{\psi}_3 = \omega \quad (4)$$

where ω is a constant.

Equations (2) and (3) are not yet in final form. Substitutions must be made from Eqs. (1) and (4), and I_2 must be replaced with I_1 (or vice versa). Furthermore, in order to study the solutions of these equations, it is convenient to introduce a new independent variable τ by means of the definition

$$\tau = pt$$

and to replace the two second-order equations (2) and (3) with four first-order equations, which is accomplished by defining four new dependent variables $x_1, \ldots, x_4$ as

$$x_1(\tau) = \psi_1 \qquad x_2(\tau) = \psi_2$$
$$x_3(\tau) = d\psi_1/d\tau \quad x_4(\tau) = d\psi_2/d\tau$$

The resulting four equations then can be expressed in matrix form as

$$(d/d\tau)x(\tau) = W(\tau)x(\tau) \quad (5)$$

where $x(\tau)$ denotes the column matrix whose elements are $x_1(\tau), \ldots, x_4(\tau)$, and $W(\tau)$ represents the square matrix

$$W(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ F_1 & F_2 & F_3 & F_4 \\ -F_2 & F_5 & -F_4 & F_3 \end{bmatrix} \quad (6)$$

with

$$F_1 = 1 - \frac{[1 + (\omega/\Omega)](I_3/I_1)}{\beta}\frac{\Omega^2}{p^2}$$
$$F_2 = (\gamma/\beta)(\Omega/p)$$
$$F_3 = -(\gamma/\beta)$$
$$F_4 = 2 - \frac{[1 + (\omega/\Omega)](I_3/I_1)}{\beta}\frac{\Omega}{p} \quad (7)$$
$$F_5 = 4 - \frac{[4 + (\omega/\Omega)](I_3/I_1)}{\beta}\frac{\Omega^2}{p^2}$$

and

$$\beta = 1 + 2(mB^2/I_1)[(A/B) + \sin\tau]^2$$
$$\gamma = 4(mB^2/I_1)[(A/B)\cos\tau + \tfrac{1}{2}\sin 2\tau] \quad (8)$$

Note that W is a periodic matrix of period 2π and that this matrix depends on five (dimensionless) parameters, I_3/I_1, ω/Ω, A/B, mB^2/I_1, and Ω/p, the first two of which depend, in turn, solely on the constitution and motion of the rigid body R.

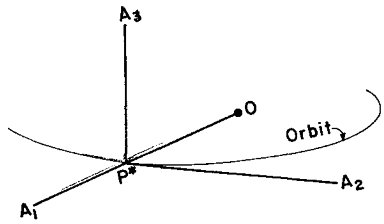

Fig. 2 Reference axes

Stability

In accordance with generalized Floquet theory,[3] the boundedness of the solutions of Eqs. (5) depends on the value at 2π of the 4×4 matrix $H(\tau)$ defined by the matrix differential equations

$$[d/d(\tau)]H(\tau) = W(\tau)H(\tau) \quad (9)$$

and the initial conditions

$$H(0) = I \quad (10)$$

where I is the 4×4 unit matrix. Specifically, all solutions of Eqs. (5) are bounded as $\tau \to \infty$ if and only if the modulus of each of the four characteristic values of $H(2\pi)$ is less than or equal to unity, and if, for any characteristic value λ_k such that $|\lambda_k| = 1$, the multiplicity of λ_k is equal to the nullity of the matrix $H(2\pi) - \lambda_k I$. To determine whether or not a given motion of the rigid body R is stable in the sense that ψ_1 and ψ_2 can be kept arbitrarily small by appropriate (nonzero) choice of $\psi_1, \psi_2, \dot{\psi}_1, \dot{\psi}_2$ at $t = 0$, one thus proceeds as follows:

1) Evaluate the parameters I_3/I_1, ω/Ω, A/B, mB^2/I_1, and Ω/p.

2) Use Eqs. (6–8) to form $W(\tau)$.

3) With the initial values specified by Eqs. (10), perform a numerical (digital computer) integration of the 16 simultaneous first-order differential equations (9) in the interval $0 \leq \tau \leq 2\pi$, thus obtaining the 16 elements of the matrix $H(2\pi)$.

4) Find the four roots $\lambda_1, \ldots, \lambda_4$ of the characteristic equation $\det[H(2\pi) - \lambda I] = 0$.

5) Evaluate $\mathrm{mod}\lambda_k$, $k = 1, \ldots, 4$, and compare each of these with unity.

This procedure, which involves some rather arduous computations, is necessary only because the differential equations of motion of R have variable coefficients, these arising from the prescribed motions of the particles P and P'. In the absence of the particles, far simpler criteria can be used to test the stability of a given motion of R. These have been discussed by Thomson[4] and Kane, Marsh, and Wilson,[5] and presently there will be occasion to refer to them.

Numerical Example

To show that attitude stabilization can be achieved by the method under consideration, take $I_3/I_1 = 1.5$ and $\omega/\Omega = -1$. These values are chosen for the following reason: $\omega/\Omega = -1$ corresponds to a motion during which the attitude of the satellite in inertial space remains essentially fixed, which is desirable for a number of practical applications, e.g., an orbiting astronomical observatory; and, in the absence of the particles P and P', this motion is unstable for $I_3/I_1 = 1.5$, as may be verified by reference to the stability chart in Ref. 5 or to Thomson's conditions,[4] the second of which is not satisfied in this case.

If the parameters that characterize P and P' now have the values $A/B = 10$, $mB^2/I_1 = 10 \times 10^{-4}$, and $\Omega/p = 0.1$, the procedure described in the preceding section leads to the

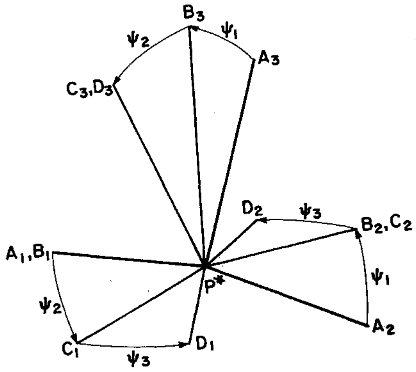

Fig. 3 Attitude angles

conclusion that the corresponding motion is stable.‡

Stabilization can be accomplished with relatively light particles performing oscillations of relatively small amplitude and low frequency, all of which points in the direction of practical feasibility. For example, if R is a homogeneous, right-circular cylinder of mass M, with equal radius and height of 5 ft, then R can be stabilized (for motion with $\omega/\Omega = -1$) by particles of mass $m = 0.012M$ oscillating with an amplitude $B = 10$ in. and with a circular frequency $p = 10\Omega$.

‡ The necessary computations were performed on a Burroughs 220 computer, generously made available to the authors by the Computation Center of Stanford University.

Conclusion

It has been shown that attitude stabilization by means of the proposed mechanism is possible in principle and may be feasible in practice. Actual design, and particularly optimum design, of a satellite stabilized in this fashion would require repeated use of the procedure described earlier and would be facilitated by prior exploration of the five-dimensional parameter space of the problem. Such an exploration is in progress and may be the subject of a later paper.

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Optimal Low-Thrust Near-Circular Orbital Transfer

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Optimum low-thrust transfers between neighboring coplanar circular orbits have been studied for a simplified system model. Small deviations from an original circular orbit motion are assumed, and the thrust acceleration is taken to be constant. Within the limits of these assumptions, the numerical results are sufficient to describe in general the optimum thrust steering program and performance capability of a vehicle in terms of its thrust/weight ratio, orbital frequency, and thrust duration. For values of thrust duration equal to an integral multiple of the orbital period, the optimum thrust direction is continuously circumferential.

Nomenclature

$a_{1,2,3,4}$ = functions of τ defined by Eqs. (17)
A,B = functions of τ defined by Eqs. (32) and (33)
$b_{1,2,3,4}$ = functions of τ defined by Eqs. (17)
C = $(A^2 + B^2)^{1/2}$
F = integrand of I
I = functional to be minimized
ΔI = total variation of I from its minimum value
J_1,J_2 = integrals required to vanish for terminal conditions satisfying circular orbital motion
m = mass of the vehicle
P = const = $T/m\omega_0^2$
r = radial distance of the vehicle from the center of the earth
t = time
T = thrust of the vehicle
u = $dx/d\tau$
v = $dy/d\tau$
x,y = position components of the vehicle defined in Fig. 1
θ = thrust steering angle defined in Fig. 1
$\delta\theta$ = variation of θ from its optimum value
Λ_1,Λ_2 = constant Lagrange multipliers
μ = gravitational parameter of the earth
τ = nondimensional time = $\omega_0 t$
τ_1 = arbitrary nondimensional time, $0 \leq \tau_1 \leq \tau$
ω = circular orbital frequency of the vehicle

Subscripts

0 = initial values at $\tau = 0$
f = final values at $\tau = \tau_f$

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Introduction

THE problem of determining optimum rocket trajectories by the indirect methods of the calculus of variations has received considerable attention in the past decade. The well-known bilinear tangent thrust steering program for optimum boost maneuvers in a vacuum and a constant gravitational field has been covered amply in the literature.[1–9]